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VASSILIEV INVARIANTS FOR TORUS KNOTS

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ABSTRACT

Vassiliev invariants up to order six for arbitrary torus knots $\{n, m\}$, with n and m coprime integers, are computed. These invariants are polynomials in n and m whose degree coincide with their order. Furthermore, they turn out to be integer-valued in a normalization previously proposed by the authors.

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1. Introduction

Vassiliev invariants [1,2,3] seem to be a very promising set of knot invariants to classify knot types. Since the discovery of their formulation in terms of inductive relations for singular knots [4,5], and of their relation to knot invariants based on quantum groups or in Chern-Simons gauge theory [6,7,4,8,5], several works have been performed to analyze Vassiliev invariants in both frameworks [9,10,11,12,13]. In [9,10] it was shown that Vassiliev invariants can be understood in terms of representations of chord diagrams whithout isolated chords modulo the so called 4T relations (weight systems), and that using semi-simple Lie algebras weight systems can be constructed. It was also shown in [10], using Kontsevitch's representation for Vassiliev invariants [14], that the space of weight systems is the same as the space of Vassiliev invariants. In [11] it was shown that these representations are precisely the ones underlying quantum-group or Chern-Simons invariants.

Recently, we made the observation [12] that the generalization of the integral or geometrical knot invariant first proposed in [15] and further analyzed in [7], as well as the invariant itself are Vassiliev invariants. In [12] we proposed an organization of those geometrical invariants and we described a procedure for their calculation from known polynomial knot invariants. This procedure was applied to obtain Vassiliev knot invariants up to order six for all prime knots up to six crossings. These geometrical invariants have also been studied by Bott and Taubes [16] using a different approach. In this paper we use the techniques used in [12] to compute all Vassiliev invariants up to order six for arbitrary torus knots. Torus knots are labelled by two coprime integers n and m, $(n,m)=1$, such that the torus knots $\{n,m\}$, $\{m,n\}$, $\{-n,-m\}$ and $\{-m,-n\}$ are the same knot, and $\{n,m\}$ and $\{n,-m\}$ are mirror images of each other. The resulting invariants are polynomials in n and m. This is consistent with the characterization found in [17,18] for torus knots of the form $n=2$ and $m=2p+1$. These polynomials turn out to be integer-valued when $(n,m)=1$ after they are normalized using the normalization proposed in [12].

The paper is organized as follows. In sect. 2 we give a brief description of the framework proposed in [12] for Vassiliev invariants. In sect. 3 we collect known results on polynomials invariants for torus knots associated to different groups and representations which will be used in our computations. In sect. 4 we calculate the Vassiliev invariants for torus knots up to order 6. Finally, in sect. 5 we analyze the properties of these invariants. An appendix deals with some technical details.

2. Vassiliev Invariants from Chern-Simons Gauge Theory

In this section we will recall a few facts on Chern-Simons gauge theory and we will summarize the formulation for Vassiliev invariants based on Chern-Simons perturbation theory proposed in [12]. Let us consider a gauge group G and a connection A on $\mathbf{R^3}$. The Chern-Simons action is defined as:

$$S_k(A) = \frac{k}{4\pi} \int_{\mathbf{R^3}} \mathrm{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{2.1}$$

where "Tr" denotes the trace in the fundamental representation of G. Given a knot C, *i.e.*, an embedding of S^1 into $\mathbf{R^3}$, we define the Wilson line associated to C carrying a representation R of G as:

$$W_C^R = \mathrm{Tr}_R \big[\mathrm{P} \exp \oint A \big], \tag{2.2}$$

where "P" stands for path ordered and the trace is to be taken in the representation R. The vacuum expectation value of W_C^R is defined as the following ratio of functional integrals:

$$\langle W_C^R \rangle = \frac{1}{Z_k} \int [DA] W_C^R \mathrm{e}^{iS_k(A)}, \tag{2.3}$$

where Z_k is the partition function:

$$Z_k = \int [DA] \mathrm{e}^{iS_k(A)}. \tag{2.4}$$

The theory based on the action (2.1) possesses a gauge symmetry which has to be fixed. In addition, one has to take into account that the theory must be regularized due to the presence of divergent integrals when performing the perturbative expansion of (2.3). Regarding these two problems we will follow the approach taken in [12]. Namely, we will work in the Landau gauge and we will assume that there exist a regularization such that one can ignore the shift in k, $k \to k + g^\vee$ (being $g^\vee$ the dual coxeter number of G), and Feynman diagrams which contain higher-loop contributions to two and three-point functions (we refer the reader to [12] for the details concerning this issue).

There is one more problem emanating from perturbation quantum field theory which must be considered. Often, products of operators $A_\mu(x)A_\nu(y)$ must be considered at the same point $x = y$, where they are divergent. This leads to an ambiguity which is solved guided by the topological nature of the theory. In the process one needs to introduce a framing attached to the knot which is characterized by an integer q. It was shown in [19] that to work in the standard framing ($q = 0$) is equivalent to ignore diagrams containing collapsible propagators in the sense explained in [19,12]. As in [12] we will indeed work in the standard framing.

As shown in [12] the perturbative expansion of the vacuum expectation value of the Wilson line (2.2) has the form:

$$\langle W_C^R \rangle = d(R) \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij} r_{ij} x^i, \tag{2.5}$$

where $x = \frac{2\pi i}{k}$, and $d(R)$ is the dimension of the representation R. The factors α_{ij} and r_{ij} in (2.5) incorporate all dependence dictated from the Feynman rules apart from the dependence on k which is contained in x. The power of x, i, represents the order in perturbation theory. Of the two factors, r_{ij} and α_{ij}, the first one contains all the group-theoretical dependence, while the second all the geometrical dependence. The quantity d_i denotes the number of independent group structures r_{ij} which appear at order i. The first values of d_i, α_{ij} and r_{ij} are: $\alpha_{0,1} = r_{0,1} = 1$, $d_0 = 1$ and $d_1 = 0$. Notice that we are dispensing with the shift in k ($x = \frac{2\pi i}{k}$), and therefore no diagrams with higher-loop contributions to two and three-point functions should be considered. In addition, there is no linear term in the expansion ($d_1 = 0$) so that diagrams with collapsible propagators should be ignored in the sense explained in [12]. It was proven in [12] that the quantities α_{ij} are Vassiliev invariants of order i.

The group factors up to order $i = 6$ have been computed in [12] in terms of the basic Casimir invariants. These group factors are classified in two types, the ones which are not products of lower order group factors:

$$
\begin{aligned}
r_{2,1} &= \sum_{l=1}^{M} C_3^{(l)}, & r_{5,4} &= \sum_{l=1}^{M} C_5^{(l)}, \\
r_{3,1} &= \sum_{l=1}^{M} \big(C_3^{(l)}\big)^2 \big(C_2^{(l)}\big)^{-1}, & r_{6,5} &= \sum_{l=1}^{M} \big(C_3^{(l)}\big)^5 \big(C_2^{(l)}\big)^{-4}, \\
r_{4,2} &= \sum_{l=1}^{M} \big(C_3^{(l)}\big)^3 \big(C_2^{(l)}\big)^{-2}, & r_{6,6} &= \sum_{l=1}^{M} C_4^{(l)} \big(C_3^{(l)}\big)^2 \big(C_2^{(l)}\big)^{-2}, \\
r_{4,3} &= \sum_{l=1}^{M} C_4^{(l)}, & r_{6,7} &= \sum_{l=1}^{M} C_5^{(l)} C_3^{(l)} \big(C_2^{(l)}\big)^{-1}, \\
r_{5,2} &= \sum_{l=1}^{M} \big(C_3^{(l)}\big)^4 \big(C_2^{(l)}\big)^{-3}, & r_{6,8} &= \sum_{l=1}^{M} C_6^{1\,(l)}, \\
r_{5,3} &= \sum_{l=1}^{M} C_4^{(l)} C_3^{(l)} \big(C_2^{(l)}\big)^{-1}, & r_{6,9} &= \sum_{l=1}^{M} C_6^{2\,(l)},
\end{aligned}
\tag{2.6}
$$

and the ones which can be written as products of lower order ones:

$$
\begin{aligned}
r_{4,1} &= r_{2,1}^2, & r_{6,2} &= r_{3,1}^2, \\
r_{5,1} &= r_{2,1} r_{3,1}, & r_{6,3} &= r_{2,1} r_{4,2}, \\
r_{6,1} &= r_{2,1}^3, & r_{6,4} &= r_{2,1} r_{4,3}.
\end{aligned}
\tag{2.7}
$$

We will refer to the group factors in (2.6) as primitive group factors, and we will denote by $\tilde{d}_i$ the number of primitive group factors at order i. From (2.6) and

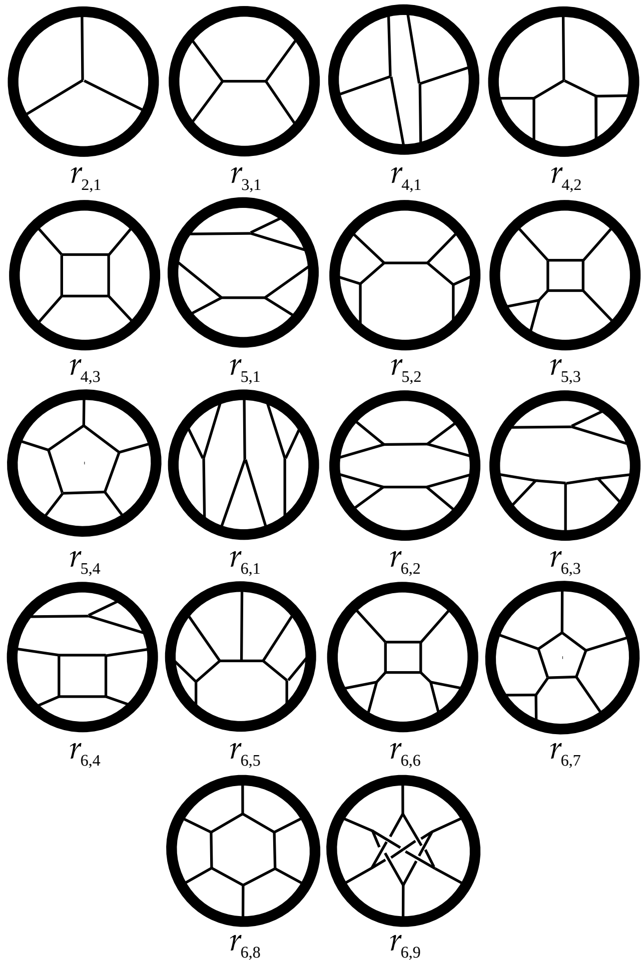

Fig. 1. Diagrams associated to the independent group factors.

(2.7) follow that the values of d_i and $\tilde{d}_i$ for $i = 1$ to 6 are:

$$\begin{aligned} d =& 0, \quad 1, \quad 1, \quad 3, \quad 4, \quad 9, \\ \tilde{d} =& 0, \quad 1, \quad 1, \quad 2, \quad 3, \quad 5. \end{aligned} \tag{2.8}$$

These dimensions are known up to $i = 9$ [10]. The notation used in (2.6) is the following. We will assume that the semi-simple group G has the form $G = \otimes_{l=1}^{n} G^{(l)}$ where $G^{(l)}$, $l = 1, \ldots, M$, are simple ones. The quantities $C_2^{(l)}$, $C_3^{(l)}$, $C_4^{(l)}$, $C_5^{(l)}$, $C_6^{1(l)}$ and $C_6^{2(l)}$ are traces of Casimir operators which for any simple group H have the form (we remove the superindex (l)):

$$
\begin{aligned}
C_2 d(R) &= \mathrm{Tr}(T_a T_a), \\
C_3 d(R) &= -f_{abc} \mathrm{Tr}(T_a T_b T_c), \\
C_4 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dsp} \mathrm{Tr}(T_a T_b T_c T_d), \\
C_5 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dst} f_{etp} \mathrm{Tr}(T_a T_b T_c T_d T_e), \\
C_6^1 d(R) &= f_{apq} f_{bqr} f_{crs} f_{dst} f_{etu} f_{rup} \mathrm{Tr}(T_a T_b T_c T_d T_e T_r), \\
C_6^2 d(R) &= f_{apq} f_{brs} f_{ctp} f_{dur} f_{eqs} f_{gtu} \mathrm{Tr}(T_a T_b T_c T_d T_e T_g),
\end{aligned}
\tag{2.9}
$$

where T_a, $a = 1, \ldots, \dim(H)$, are the generators of the simple group H in the representation R, and f_{abc} are the structure constants. The diagrams associated to these Casimirs are presented in Fig. 2 while the diagrams associated to the groups factors (2.6) and (2.7) are contained in Fig. 1.

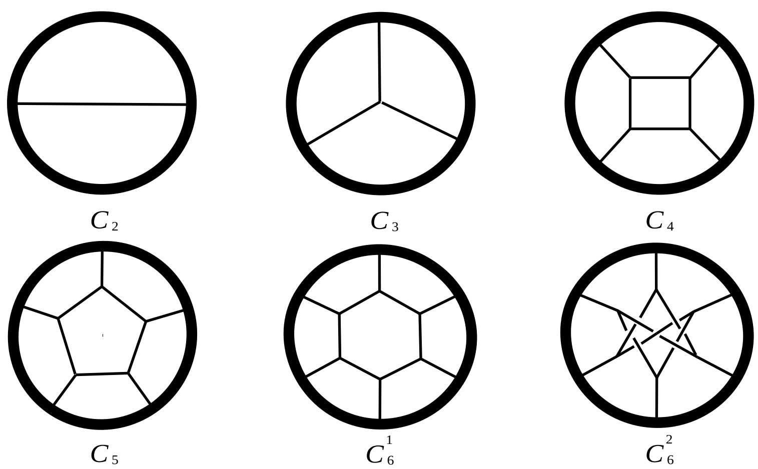

Fig. 2. Diagramas associated to Casimir operators.

As in [12] we will use information from knot invariants for $SU(N)$ and $SO(N)$ in the fundamental representation, and for $SU(2)$ in an arbitrary representation of spin $j/2$. The form of the Casimirs (2.9) for these cases, taken in a normalization such that for the fundamental representation,

$$\mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}, \tag{2.10}$$

is [12]:

- $SU(N)_f$:

$$\begin{aligned} C_2 =& -\frac{1}{2N}(N^2-1), \\ C_3 =& -\frac{1}{4}(N^2-1), \\ C_4 =& \frac{1}{16}(N^2-1)(N^2+2), \\ C_5 =& \frac{1}{32}N(N^2-1)(N^2+1), \\ C_6^1 =& \frac{1}{64}(N^2-1)(N^4+N^2+2), \\ C_6^2 =& \frac{1}{64}(N^2-1)(3N^2-2); \end{aligned} \tag{2.11}$$

- $SO(N)_f$:

$$\begin{aligned} C_2 =& -\frac{1}{4}(N-1), \\ C_3 =& -\frac{1}{16}(N-1)(N-2), \\ C_4 =& \frac{1}{256}(N-1)(N-2)(N^2-5N+10), \\ C_5 =& \frac{1}{1024}(N-1)(N-2)(N^3-7N^2+17N-10), \\ C_6^1 =& \frac{1}{4096}(N-1)(N-2)(N^2-7N+14)(N^2-2N+3), \\ C_6^2 =& \frac{1}{4096}(N-1)(N-2)(N-3)(7N-18); \end{aligned} \tag{2.12}$$

• $SU(2)_j$:

$$\begin{aligned}
C_2 =& - j(j+1), \\
C_3 =& - j(j+1), \\
C_4 =& 2j^2(j+1)^2, \\
C_5 =& 3j^2(j+1)^2 - j(j+1), \\
C_6^1 =& 2j^3(j+1)^3 + 3j^2(j+1)^2 - 2j(j+1), \\
C_6^2 =& - 2j^3(j+1)^3 + 5j^2(j+1)^2 - 2j(j+1).
\end{aligned} \tag{2.13}$$

Equations (2.6), (2.7) and (2.9) contain all the information appearing on the right hand side of (2.5) up to order six except the Vassiliev invariants themselves (the quantities α_{ij}) which we intend to compute. To solve for them we need to know the left hand side of (2.5). This information is compiled in next section.

3. HOMFLY, Kauffman, Jones, and Akutsu-Wadati Polynomials for Torus Knots

In this section we convenientely present the form of the polynomial invariants for torus knots corresponding to the groups $SU(N)$ and $SO(N)$ in the fundamental representation, and to the group $SU(2)$ in an arbitrary representation of spin $j/2$. These quantities have been worked out from different points of view in the last few years.

The polynomial invariant for $SU(N)$ in the fundamental representation is the HOMFLY polynomial [20,22,21]. It was first computed for torus knots in [21], reobtained from quantum groups in [23], and from Chern-Simons gauge theory in [24]. It has the form:

$$\begin{aligned}
\tilde{P}_{n,m}((\lambda t)^{\frac{1}{2}}, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) =& \Big(\frac{1-t}{1-t^n}\Big)\frac{\lambda^{\frac{1}{2}(m-1)(n-1)}}{\lambda t - 1} \\
& \times \sum_{\substack{p+i+1=n \\ p,i\geq 0}} (-1)^i t^{mi+\frac{1}{2}p(p+1)} \frac{\prod_{j=-p}^{i}(\lambda t - t^j)}{(i)!(p)!}
\end{aligned} \tag{3.1}$$

where $(x) = t^x - 1$, $\lambda = t^{N-1}$, and $t = \exp(2\pi i/(k+g^\vee))$ with $g^\vee = N$.

The polynomial invariant for $SO(N)$ in the fundamental representation corresponds to the Kauffman polynomial [25]. It was first computed for torus knots in [26] and reobtained from Chern-Simons gauge theory in [27]. It has the following form:

$$\tilde{F}_{n,m}(\lambda, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \frac{[1]\,\lambda^{nm}}{[1]+[0;1]} \times \left(\sum_{\substack{\gamma+\beta+1=n \\ \gamma,\beta\geq 0}} t^{-\frac{m}{2}(\beta-\gamma)} \lambda^{-m} (-1)^{\gamma} \times \left(\frac{1}{[n]} + \frac{1}{[\beta-\gamma;1]} \right) \right.$$
$$\left. \times \frac{1}{[\beta]!\ [\gamma]!} \times \prod_{j=-\gamma}^{\beta} [j;1] + \begin{cases} 0 & \text{n odd} \\ 1 & \text{n even} \end{cases} \right), \tag{3.2}$$

where $[p] = t^{\frac{p}{2}} - t^{-\frac{p}{2}}$, $[p;q] = t^{\frac{p}{2}}\lambda^{q} - t^{-\frac{p}{2}}\lambda^{-q}$, $\lambda = t^{\frac{N-1}{2}}$, and $t = \exp(\pi i/(k+g^\vee))$ with $g^\vee = N-2$.

The polynomial invariant for $SU(2)$ in an arbitrary representation of spin $j/2$ is known as the Jones polynomial [20] for $j = 1$, and the Akutsu-Wadati polynomial for $j > 1$. Its form for torus knots was first obtained in [28]. The framework utilized was based on the formalism proposed in [29]. Later it was reobtained in [30] also in the framework of Chern-Simons gauge theory but using a different formalism. This invariant has the form:

$$\tilde{I}^{j}_{n,m}(t) = \frac{t^{\frac{j}{2}(n-1)(m-1)}}{t^{j+1}-1} \sum_{l=0}^{j} t^{n(1+ml)(j-l)} (t^{1+sl} - t^{m(j-l)}), \tag{3.3}$$

where $t = \exp(2\pi i/(k+2))$.

The invariants listed in (3.1), (3.2) and (3.3) contain enough information to obtain the quantities α_{ij} in (2.5) up to order six. Actually, we must take into account that in (3.1), (3.2) and (3.3) the invariants have been normalized in such a way that for the unknot they take the value one. In the expression (2.5) we need however the unnormalized invariants, *i.e.*, the expressions for the Wilson lines as obtained from (2.3). These are easily obtained multiplying by the value of the

Wilson lines corresponding to the unknot as dictated from Chern-Simons gauge theory. These expressions can be extracted, for example, from [31] and [32] for the cases of $SU(N)$ and $SO(N)$ in the fundamental representation respectively, and from [28] for $SU(2)$ in a representation of spin $j/2$. Taking into account these facts it turns out that for the left-hand side of (2.5) we must consider:

$$P_{n,m}((\lambda t)^{\frac{1}{2}}, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = \frac{(\lambda t)^{\frac{1}{2}} - (\lambda t)^{-\frac{1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \tilde{P}_{n,m}((\lambda t)^{\frac{1}{2}}, t^{\frac{1}{2}} - t^{-\frac{1}{2}}), \tag{3.4}$$

where $(x) = t^x - 1$, $\lambda = t^{N-1}$ and $t = \exp(2\pi i/(k+N))$,

$$F_{n,m}(\lambda, t^{\frac{1}{2}} - t^{-\frac{1}{2}}) = (1 + \frac{\lambda - \lambda^{-1}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}) \tilde{F}_{n,m}(\lambda, t^{\frac{1}{2}} - t^{-\frac{1}{2}}), \tag{3.5}$$

where $\lambda = t^{\frac{N-1}{2}}$ and $t = \exp(\pi i/(k+N-2))$, and

$$I^j_{n,m}(t) = \frac{t^{\frac{j+1}{2}} - t^{-\frac{j+1}{2}}}{t^{\frac{1}{2}} - t^{-\frac{1}{2}}} \tilde{I}^j_{n,m}(t), \tag{3.6}$$

where $t = \exp(2\pi i/(k+2))$.

To obtain the linear equations satisfied by the α_{ij} one must appropriately expand the invariants (3.4), (3.5) and (3.6). According to the form of t entering in each of them, and taking into account that one is dispensing with the shift, one has to introduce the variable x through:

$$\begin{aligned} t =& \mathrm{e}^{x}, \qquad \text{for} \;\; SU(N), \\ t =& \mathrm{e}^{\frac{x}{2}}, \qquad \text{for} \;\; SO(N). \end{aligned} \tag{3.7}$$

4. Vassiliev Invariants for Torus Knots up to order six

In this section we obtain the expression of the Vassiliev invariants α_{ij} in (2.5) up to order six. To carry this out one must solve the equations resulting after considering the power series expansion of the left hand side of (2.5) once the variable x has been introduced as dictated by t in (3.7).

In expanding (3.4) and (3.5) one finds the slight complication that n appears in the limits of the sum and the product entering those expressions. A similar complication shows up in (3.6) due to the presence of a sum whose upper limit is j. To solve these difficulties we will proceed in the following way. First notice that the invariants $P_{n,m}$, $F_{n,m}$ and $I_{n,m}$ are symmetric in n and m. This implies that the coefficient of the Taylor series expansion in x must also be symmetric in n and m. On the other hand, for n fixed in (3.4) and (3.5), and for j fixed in (3.6), it is clear that the coefficients are polynomials in m, and in n and m respectively. Actually, it was shown in [17,18] that if $n = 2$ and $m = 2p + 1$ the polynomial in p resulting as the coefficient of x^i is at most a polynomial of degree i. We can therefore assume that the coefficient of x^i in the Taylor series expansion of (3.4), (3.2) and (3.6) is a symmetric polynomial in n and m of degrees i in both n and m.

One can actually state one more property of the polynomials in n and m entering the coefficients of the Taylor series expansion of (3.4), (3.5) and (3.6). Under an inversion of space, a torus knot $\{n, m\}$ becomes the torus knot $n, -m$. On the other hand, it is known that HOMFLY, Kauffman and Akutsu-Wadati invariants which correspond to knots which are mirror images of each other are related by the transformation $t \to t^{-1}$. This implies that the coefficients of the Taylor series expansions of (3.4), (3.5) and (3.6) are invariant under the change $m \to -m$ (or $n \to -n$) and $x \to -x$. The consequences of this symmetry are the following: the polynomial of the coefficient of x^i with i even contains only even powers of n and m; the polynomial of the coefficient of x^i with i odd is of the form nm times a polynomial containing only even powers of n and m.

The properties discussed so far are satisfied by both the unnormalized invariants (3.4), (3.5) and (3.6), and the normalized ones (3.1), (3.2) and (3.3). There is, however, one important property shared by the normalized invariants which do not have the unnormalized ones and that suggests that one should analyze first those. One can verify from (3.1), (3.2) and (3.3) that $P_{1,\pm m} = F_{1,\pm m} = I^j_{1,\pm m} = 1$. This implies that ± 1 must be roots of all the polynomials entering the coefficients of x^i for $i > 0$ in both, n and m, in the corresponding power series. To exploit this property, instead of considering the expansion (2.5) for the unormalized Wilson lines we will consider the corresponding expansion for the normalized ones,

$$\langle \tilde{W}^R_C \rangle = \frac{\langle W^R_C \rangle}{\langle W^R_0 \rangle} = \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \tilde{\alpha}_{ij} r_{ij} x^i, \tag{4.1}$$

where $\langle W^R_0 \rangle$ denotes the vacuum expectation value for the unknot in the representation R. Notice that we are using the same notation as in [12]. We will compute the quantities $\tilde{\alpha}_{ij}$ up to order six. Clearly, from that result plus the fact the for the unknot [12]:

$$\begin{aligned}\langle W^R_0 \rangle = \; d(R) \Big(& \frac{r_{1,2}}{6} \, x^2 + (\frac{{r_{1,2}}^2}{72} + \frac{r_{2,4}}{360} - \frac{r_{3,4}}{360}) \, x^4 + (\frac{-{r_{1,2}}^3}{1296} - \frac{r_{1,2} \, r_{2,4}}{2160} \\ & + \frac{r_{1,2} \, r_{3,4}}{2160} - \frac{r_{5,6}}{15120} + \frac{r_{6,6}}{3780} - \frac{r_{7,6}}{11340} + \frac{r_{8,6}}{9072} - \frac{r_{9,6}}{15120}) \, x^6 + O(x^8) \Big),\end{aligned} \tag{4.2}$$

one easily obtains the expression for the α_{ij}.

From the properties described in the previous paragraphs follow that if i is even, the polynomial corresponding to the coefficient of x^i in the Taylor series expansion of (3.1), (3.2) and (3.3) has the form $(n^2-1)(m^2-1)$ times a polynomial in m^2 and n^2 which is symmetric under $m^2 \leftrightarrow n^2$, and possesses at most degree $i-2$ in both n and m. For i odd the corresponding polynomial has the form $nm(n^2-1)(m^2-1)$ times a polynomial in m^2 and n^2 which is symmetric under $m^2 \leftrightarrow n^2$, and possesses at most degree $i-3$ in both n and m.

Up to order $i = 6$ the general form of the Taylor series expansions of (3.1), (3.2) and (3.3) becomes:

$$\begin{aligned}
&1 + (n^2-1)\,(m^2-1)\,x^2\,g_{2,1} + n\,m\,(n^2-1)\,(m^2-1)\,x^3\,g_{3,1} \\
&+ (n^2-1)\,(m^2-1)\,x^4\,(g_{4,1} + (n^2+m^2)\,g_{4,2} + n^2\,m^2\,g_{4,3}) \\
&+ n\,m\,(n^2-1)\,(m^2-1)\,x^5\,(g_{5,1} + (n^2+m^2)\,g_{5,2} + n^2\,m^2\,g_{5,3}) \\
&+ (n^2-1)\,(m^2-1)\,x^6\,(g_{6,1} + (n^2+m^2)\,g_{6,2} + n^2\,m^2\,g_{6,3} \\
&\qquad + (n^4\,m^2 + n^2\,m^4)\,g_{6,4} + (n^4+m^4)\,g_{6,5} + n^4\,m^4\,g_{6,6}),
\end{aligned} \tag{4.3}$$

where g_{ij} are functions of N for the cases of the HOMFLY and the Kauffman invariants, and functions of the spin $j/2$ for the case of Akutsu-Wadati. The simplest way to determine these functions is to generate linear relations among them computing the Taylor series expansions of the left-hand side of (4.1) for specific values of n and m up to order six. In doing this one finds the following results:

- $SU(N)_f$:

$$\begin{aligned}
g_{2,1} &= -\frac{1}{24}(N^2-1), & g_{4,3} &= -\frac{1}{1920}(N^4-1), \\
g_{3,1} &= -\frac{1}{144}N(N^2-1), & g_{5,1} &= \frac{1}{86400}N(29N^4-60N^2+31), \\
g_{4,1} &= \frac{1}{5760}(7N^4-10N^2+3), & g_{5,2} &= -\frac{1}{86400}N(11N^4-40N^2+29), \\
g_{4,2} &= -\frac{1}{5760}(3N^4-10N^2+7), & g_{5,3} &= -\frac{1}{86400}N(N^4-10N+11),
\end{aligned}$$

$$
\begin{aligned}
g_{6,1} =& -\frac{1}{967680}(31N^6-49N^4+21N^2-3),\\
g_{6,2} =& \frac{1}{483840}(9N^6-35N^4+35N^2-9),\\
g_{6,3} =& \frac{1}{1451520}(55N^6-98N^4-35N^2+78),\\
g_{6,4} =& -\frac{1}{1451520}(22N^6-77N^4+28N^2+27),\\
g_{6,5} =& -\frac{1}{967680}(3N^6-21N^4+49N^2-31),\\
g_{6,6} =& \frac{1}{2903040}(5N^6-49N^4+35N^2+9);
\end{aligned}
\tag{4.4}
$$

- $SO(N)_f$:

$$
\begin{aligned}
g_{2,1} =& -\frac{1}{96}(N^2-3N+2),\\
g_{3,1} =& -\frac{1}{1152}(N-2)^2(N-1),\\
g_{4,1} =& \frac{1}{92160}(7N^4-45N^3+110N^2-120N+48),\\
g_{4,2} =& -\frac{1}{92160}(3N^4-15N^3+20N^2-8),
\end{aligned}
$$

$$
\begin{aligned}
g_{4,3} =& -\frac{1}{92160}(3N^4-25N^3+70N^2-80N+32),\\
g_{5,1} =& \frac{1}{5529600}(58N^5-469N^4+1455N^3-2120N^2+1412N-336),\\
g_{5,2} =& -\frac{1}{5529600}(22N^5-141N^4+295N^3-180N^2-92N+96),\\
g_{5,3} =& -\frac{1}{5529600}(2N^5-51N^4+245N^3-480N^2+428N-144),
\end{aligned}
$$

$$
\begin{aligned}
g_{6,1} =& -\frac{1}{61931520}(31N^6 - 315N^5 + 1358N^4 - 3150N^3 + 4116N^2 \\
& \qquad - 2856N + 816), \\
g_{6,2} =& \frac{1}{61931520}(18N^6 - 147N^5 + 455N^4 - 630N^3 + 280N^2 + 168N - 144), \\
g_{6,3} =& \frac{1}{928972800}(550N^6 - 5768N^5 + 23443N^4 - 46865N^3 + 47740N^2 \\
& \qquad - 22652N + 3552), \\
g_{6,4} =& -\frac{1}{928972800}(220N^6 - 1757N^5 + 5152N^4 - 6335N^3 + 1540N^2 \\
& \qquad + 3052N - 1872), \\
g_{6,5} =& -\frac{1}{61931520}(3N^6 - 21N^5 + 42N^4 - 56N^2 + 32), \\
g_{6,6} =& \frac{1}{928972800}(25N^6 + 112N^5 - 1442N^4 + 4585N^3 - 6860N^2 \\
& \qquad + 5068N - 1488);
\end{aligned}
\tag{4.5}
$$

- $SU(2)_j$:

$$
\begin{aligned}
g_{2,1} =& \frac{1}{6}A, & g_{4,3} =& \frac{1}{360}A(7A+9), \\
g_{3,1} =& \frac{1}{18}A, & g_{5,1} =& \frac{1}{1080}A(10A-1), \\
g_{4,1} =& \frac{1}{360}A(7A-1), & g_{5,2} =& -\frac{1}{1080}A(6A+3), \\
g_{4,2} =& -\frac{1}{360}A(3A+1), & g_{5,3} =& \frac{1}{1080}A(18A+15),
\end{aligned}
$$

$$
\begin{aligned}
g_{6,1} =& \frac{1}{75600}A(155A^2 - 55A^2 + 5), \\
g_{6,2} =& -\frac{1}{75600}A(90A^2 + 20A - 5), \\
g_{6,3} =& \frac{1}{75600}A(260A^2 + 358A - 9)), \\
g_{6,4} =& -\frac{1}{75600}A(90A^2 + 342A + 184), \\
g_{6,5} =& \frac{1}{75600}A(15A^2 + 15A + 5), \\
g_{6,6} =& \frac{1}{75600}A(155A^2 + 1023A + 691),
\end{aligned}
\tag{4.6}
$$

where,

$$A = -\frac{1}{4} j(j+2). \tag{4.7}$$

We have all the ingredients to compute the Vassiliev invariants for torus knots up to order six. Equations (4.4), (4.5) and (4.6) plugged in (4.3) constitute the left-hand side of (4.1). Since all but the $\tilde{\alpha}_{ij}$ is known on the right-hand side of that equation we can obtain linear relations for the $\tilde{\alpha}_{ij}$ considering different groups and representations. As in [12] we will take into account the linear relations appearing from the consideration of $SU(N)$ and $SO(N)$ in their fundamental representations, $SU(2)$ in a representation of arbitrary spin $j/2$, and $SU(N) \times SU(2)$ in a representation which is the product of the fundamental of $SU(N)$ times the one of spin $j/2$ of $SU(2)$. For this last case we will use the following property. Denoting by $\langle W_C^{R_l} \rangle$ and $\langle W_C^R \rangle$ the vacuum expectation values of Wilson lines based on the simple groups G_l where $G = \otimes_{l=1}^M G_l$, being the representation R of G a direct product of representations R_l of G_l, one has,

$$\langle W_C^R \rangle = \prod_{l=1}^{M} \langle W_C^{R_l} \rangle. \tag{4.8}$$

This follows directly from the factorization of both the partition function and the Wilson line operator [12].

Proceeding in the way described above one finds 5 equations for $\tilde{\alpha}_{2,1}$, 5 equations for $\tilde{\alpha}_{3,1}$, 12 equations for $\tilde{\alpha}_{4,1}$, $\tilde{\alpha}_{4,2}$ and $\tilde{\alpha}_{4,3}$, 15 equations for $\tilde{\alpha}_{5,1}, \cdots, \tilde{\alpha}_{5,4}$, and 20 equations for $\tilde{\alpha}_{6,1}, \cdots, \tilde{\alpha}_{6,9}$. These equations have a unique solution which takes the form:

$$\tilde{\alpha}_{2,1} = \frac{1}{6}\,(n^2-1)\,(m^2-1),$$
$$\tilde{\alpha}_{3,1} = \frac{1}{18}\,n\,m\,(n^2-1)\,(m^2-1),$$
$$\tilde{\alpha}_{4,1} = \frac{1}{72}\,(n^2-1)^2\,(m^2-1)^2,$$
$$\tilde{\alpha}_{4,2} = \frac{1}{360}\,(m^2-1)\,(n^2-1)\,(9\,n^4\,m^2-m^2-n^2-1),$$
$$\tilde{\alpha}_{4,3} = \frac{1}{360}\,(n^4-1)\,(m^4-1),$$

$$\tilde{\alpha}_{5,1} = \frac{1}{108}\,n\,m\,(n^2-1)^2\,(m^2-1)^2,$$
$$\tilde{\alpha}_{5,2} = \frac{1}{5400}\,n\,m\,(n^2-1)\,(m^2-1)\,(69\,n^2\,m^2-21\,\big(n^2+m^2\big)-11\big),$$
$$\tilde{\alpha}_{5,3} = \frac{1}{5400}\,n\,m\,(n^2-1)\,(m^2-1)\,(11\,n^2\,m^2+n^2+m^2-9),$$
$$\tilde{\alpha}_{5,4} = \frac{1}{900}\,n\,m\,(n^4-1)\,(m^4-1),$$

$$
\begin{aligned}
\tilde{\alpha}_{6,1} =& \frac{1}{1296}\,(n^2-1)^3\,(m^2-1)^3,\\
\tilde{\alpha}_{6,2} =& \frac{1}{648}\,(n^2-1)^2\,(m^2-1)^2\,n^2\,m^2,\\
\tilde{\alpha}_{6,3} =& \frac{1}{2160}\,(n^2-1)^2\,(m^2-1)^2(9\,n^2\,m^2-n^2-m^2-1),\\
\tilde{\alpha}_{6,4} =& \frac{1}{2160}\,(n^2-1)^2\,(m^2-1)^2\,(n^2+1)\,(m^2+1),\\
\tilde{\alpha}_{6,5} =& \frac{1}{75600}\,(n^2-1)\,(m^2-1)\,\big(516\,n^4\,m^4-289\,(n^2\,m^4+n^4\,m^2)\\
& \qquad -44\,n^2\,m^2+5\,(n^4+m^4)+5\,(n^2+m^2)+5\big),\\
\tilde{\alpha}_{6,6} =& \frac{1}{90720}\,(n^2-1)\,(m^2-1)\,\big(53\,n^4\,m^4-101\,(n^2\,m^4+n^4\,m^2)\\
& \qquad -115\,n^2\,m^2-24\,(n^4+m^4)-24\,(n^2+m^2)-24\big),\\
\tilde{\alpha}_{6,7} =& \frac{1}{226800}\,(n^2-1)\,(m^2-1)\,\big(419\,n^4\,m^4+209\,(n^2\,m^4+n^4\,m^2)\\
& \qquad -n^2\,m^2+20\,(n^4+m^4)+20\,(n^2+m^2)+20\big),\\
\tilde{\alpha}_{6,8} =& \frac{1}{453600}\,(n^2-1)\,(m^2-1)\,\big(13\,n^4\,m^4+13\,(n^2\,m^4+n^4\,m^2)\\
& \qquad +13\,n^2\,m^2-50\,(n^4+m^4)-50\,(n^2+m^2)-50\big),\\
\tilde{\alpha}_{6,9} =& \frac{1}{151200}\,(n^2-1)\,(m^2-1)\,\big(31\,n^4\,m^4+31\,(n^2\,m^4+n^4\,m^2)\\
& \qquad +31\,n^2\,m^2+10\,(n^4+m^4)+10\,(n^2+m^2)+10\big).
\end{aligned}
\tag{4.9}
$$

One can verify that the Vassiliev invariants which do not correspond to primitive ones satisfy the relations [12]:

$$
\begin{aligned}
\tilde{\alpha}_{4,1} =& \frac{1}{2}\tilde{\alpha}_{2,1}^2, & \tilde{\alpha}_{6,2} =& \frac{1}{2}\tilde{\alpha}_{3,1}^2,\\
\tilde{\alpha}_{5,1} =& \tilde{\alpha}_{2,1}\tilde{\alpha}_{3,1}, & \tilde{\alpha}_{6,3} =& \tilde{\alpha}_{2,1}\tilde{\alpha}_{4,2},\\
\tilde{\alpha}_{6,1} =& \frac{1}{6}\tilde{\alpha}_{2,1}^3, & \tilde{\alpha}_{6,4} =& \tilde{\alpha}_{2,1}\tilde{\alpha}_{4,3}.
\end{aligned}
\tag{4.10}
$$

As explained in [12] these relations are a consequence of the factorization property (4.8).

The quantities α_{ij}, which are the ones which have a more direct integral interpretation are easily obtained from the $\tilde{\alpha}_{ij}$ in (4.9) and the expansion for the unknot up to order six (4.2). Since (4.2) only contains even powers of x, for the invariants of odd order one has:

$$\begin{aligned} \alpha_{3,1} &= \tilde{\alpha}_{3,1}, & \alpha_{5,3} &= \tilde{\alpha}_{5,3}, \\ \alpha_{5,2} &= \tilde{\alpha}_{5,2}, & \alpha_{5,4} &= \tilde{\alpha}_{5,4}. \end{aligned} \tag{4.11}$$

As shown in [12], the relations (4.10) for the compound invariants also holds in this case:

$$\begin{aligned} \alpha_{4,1} &= \frac{1}{2}\alpha_{2,1}^2, & \alpha_{6,2} &= \frac{1}{2}\alpha_{3,1}^2, \\ \alpha_{5,1} &= \alpha_{2,1}\alpha_{3,1}, & \alpha_{6,3} &= \alpha_{2,1}\alpha_{4,2}, \\ \alpha_{6,1} &= \frac{1}{6}\alpha_{2,1}^3, & \alpha_{6,4} &= \alpha_{2,1}\alpha_{4,3}. \end{aligned} \tag{4.12}$$

For the primitive ones of even order one finds:

$$\begin{aligned} \alpha_{2,1} &= \frac{1}{6}\,(n^2\,m^2 - n^2 - m^2), \\ \alpha_{4,2} &= \frac{1}{360}\,(9\,n^4\,m^4 - 10\,(n^2\,m^4 + n^4\,m^2) + (n^4 + m^4) + 10n^2\,m^2), \\ \alpha_{4,3} &= \frac{1}{360}\,(n^4\,m^4 - n^4 - m^4), \\ \\ \alpha_{6,5} &= \frac{1}{75600}\,\big(516\,n^6\,m^6 - 805\,(n^4\,m^6 + n^6\,m^4) + 1050\,n^4\,m^4 \\ &\qquad + 294\,(n^2\,m^6 + n^6\,m^2) - 245\,(n^2\,m^4 + n^4\,m^2) \\ &\qquad - 5\,(n^6 + m^6) - 49\,n^2\,m^2\big), \\ \alpha_{6,6} &= \frac{1}{90720}\,\big(53\,n^6\,m^6 - 154\,(n^4\,m^6 + n^6\,m^4) + 140\,n^4\,m^4 \\ &\qquad + 77\,(n^2\,m^6 + n^6\,m^2) + 14\,(n^2\,m^4 + n^4\,m^2) \\ &\qquad + 24\,(n^6 + m^6) - 91\,n^2\,m^2\big), \\ \alpha_{6,7} &= \frac{1}{226800}\,\big(419\,n^6\,m^6 - 210\,(n^4\,m^6 + n^6\,m^4) - 189\,(n^2\,m^6 + n^6\,m^2) \\ &\qquad + 210\,(n^2\,m^4 + n^4\,m^2) - 20\,(n^6 + m^6) - 21\,n^2\,m^2\big), \\ \alpha_{6,8} &= \frac{1}{453600}\,\big(13\,n^6\,m^6 - 63\,(n^2\,m^6 + n^6\,m^2) + 50\,(n^6 + m^6) + 63\,n^2\,m^2\big), \\ \alpha_{6,9} &= \frac{1}{151200}\,\big(31\,n^6\,m^6 - 21\,(n^2\,m^6 + n^6\,m^2) - 10\,(n^6 + m^6) + 21\,n^2\,m^2\big). \end{aligned} \tag{4.13}$$

The simple expressions found for the quantities α_{ij} are in clear contrast with the complexity implicit in their integral form. As it was indicated above, the α_{ij} constitute the form of the Vassiliev invariants which are more simply related to their integral form as dictated from the Feynmann rules of Chern-Simons gauge theory. These are known only for the first two invariants. At order two one has [15,7]:

$$
\begin{aligned}
\alpha_{2,1} = \frac{1}{4} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int^{x_3} dx_4^{\mu_4} \Delta_{\mu_1\mu_3}(x_1 - x_3)\Delta_{\mu_2\mu_4}(x_2 - x_4) \\
-\frac{1}{16} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int_{\mathbf{R}^3} d^3y \big(\Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\nu_2}(x_2 - y) \\
\times \Delta_{\mu_3\nu_3}(x_3 - y)\epsilon^{\nu_1\nu_2\nu_3}\big),
\end{aligned}
\tag{4.14}
$$

while at order three [12]:

$$
\begin{aligned}
\alpha_{3,1} = {} & \frac{1}{8} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int^{x_3} dx_4^{\mu_4} \int^{x_4} dx_5^{\mu_5} \int^{x_5} dx_6^{\mu_6} \\
& \big[\Delta_{\mu_1\mu_4}(x_1 - x_4)\Delta_{\mu_2\mu_6}(x_2 - x_6)\Delta_{\mu_3\mu_5}(x_3 - x_5) \\
& \quad + \Delta_{\mu_1\mu_3}(x_1 - x_3)\Delta_{\mu_2\mu_5}(x_2 - x_5)\Delta_{\mu_4\mu_6}(x_4 - x_6) \\
& \quad + \Delta_{\mu_1\mu_5}(x_1 - x_5)\Delta_{\mu_2\mu_4}(x_2 - x_4)\Delta_{\mu_3\mu_6}(x_3 - x_6)\big] \\
& + \frac{1}{4} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int^{x_3} dx_4^{\mu_4} \int^{x_4} dx_5^{\mu_5} \int^{x_5} dx_6^{\mu_6} \\
& \big[\Delta_{\mu_1\mu_4}(x_1 - x_4)\Delta_{\mu_2\mu_5}(x_2 - x_5)\Delta_{\mu_3\mu_6}(x_3 - x_6)\big] \\
& - \frac{1}{32} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int^{x_3} dx_4^{\mu_4} \int^{x_4} dx_5^{\mu_5} \int_{\mathbf{R}^3} d^3y \\
& \big[\Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\mu_5}(x_2 - x_5)\Delta_{\mu_3\nu_3}(x_3 - y)\Delta_{\mu_4\nu_4}(x_4 - y)\epsilon^{\nu_1\nu_3\nu_4} \\
& \quad + \Delta_{\mu_1\mu_3}(x_1 - x_3)\Delta_{\mu_2\nu_2}(x_2 - y)\Delta_{\mu_4\nu_4}(x_4 - y)\Delta_{\mu_5\nu_5}(x_5 - y)\epsilon^{\nu_2\nu_4\nu_5} \\
& \quad + \Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\mu_4}(x_2 - x_4)\Delta_{\mu_3\nu_3}(x_3 - y)\Delta_{\mu_5\nu_5}(x_5 - y)\epsilon^{\nu_1\nu_3\nu_5} \\
& \quad + \Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\nu_2}(x_2 - y)\Delta_{\mu_3\mu_5}(x_3 - x_5)\Delta_{\mu_4\nu_5}(x_4 - y)\epsilon^{\nu_1\nu_2\nu_4} \\
& \quad + \Delta_{\mu_1\mu_4}(x_1 - x_4)\Delta_{\mu_2\nu_2}(x_2 - y)\Delta_{\mu_3\nu_3}(x_3 - y)\Delta_{\mu_5\nu_5}(x_5 - y)\epsilon^{\nu_2\nu_3\nu_5}\big] \\
& + \frac{1}{128} \oint dx_1^{\mu_1} \int^{x_1} dx_2^{\mu_2} \int^{x_2} dx_3^{\mu_3} \int^{x_3} dx_4^{\mu_4} \int_{\mathbf{R}^3} d^3y \int_{\mathbf{R}^3} d^3z \\
& \big[\big(\Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\sigma_2}(x_2 - z)\Delta_{\mu_3\sigma_3}(x_3 - z)\Delta_{\mu_4\nu_4}(x_4 - y) \\
& \qquad \times \Delta_{\alpha\beta}(y - z)\epsilon^{\nu_1\alpha\nu_4}\epsilon^{\sigma_3\beta\sigma_2}\big) \\
& \quad + \big(\Delta_{\mu_1\nu_1}(x_1 - y)\Delta_{\mu_2\nu_2}(x_2 - y)\Delta_{\mu_3\sigma_3}(x_3 - z) \\
& \qquad \times \Delta_{\mu_4\sigma_4}(x_4 - z)\Delta_{\alpha\beta}(y - z)\epsilon^{\nu_2\alpha\nu_1}\epsilon^{\sigma_4\beta\sigma_3}\big)\big],
\end{aligned}
\tag{4.15}
$$

where,

$$
\Delta_{\mu\nu}(x - y) = \frac{1}{\pi}\epsilon_{\mu\sigma\nu}\frac{(x - y)^\sigma}{|x - y|^3}.
$$

We proposed in [12] a normalization of the Vassiliev invariants which seemed to be special in the sense that these invariants become integer-valued. This normalization is fixed assuming that the Vassiliev invariants $\tilde{\alpha}_{ij}$ for the trefoil knot do not vanish and then using those non-vanishing values to normalize for the rest of the non-trivial knots inserting appropriate factors. The resulting primitive invariants for a knot K, which are labelled as $\beta_{ij}(K)$, are defined up to order six as:

$$\begin{aligned}
\beta_{2,1}(K) &= \frac{\tilde{\alpha}_{2,1}(K)}{\tilde{\alpha}_{2,1}(3_1)}, & \beta_{5,4}(K) &= \frac{\tilde{\alpha}_{5,4}(K)}{\tilde{\alpha}_{5,4}(3_1)}, \\
\beta_{3,1}(K) &= \frac{\tilde{\alpha}_{3,1}(K)}{\tilde{\alpha}_{3,1}(3_1)}, & \beta_{6,5}(K) &= 5071\frac{\tilde{\alpha}_{6,5}(K)}{\tilde{\alpha}_{6,5}(3_1)}, \\
\beta_{4,2}(K) &= 31\frac{\tilde{\alpha}_{4,2}(K)}{\tilde{\alpha}_{4,2}(3_1)}, & \beta_{6,6}(K) &= 29\frac{\tilde{\alpha}_{6,6}(K)}{\tilde{\alpha}_{6,6}(3_1)}, \\
\beta_{4,3}(K) &= 5\frac{\tilde{\alpha}_{4,3}(K)}{\tilde{\alpha}_{4,3}(3_1)}, & \beta_{6,7}(K) &= 1531\frac{\tilde{\alpha}_{6,7}(K)}{\tilde{\alpha}_{6,7}(3_1)}, \\
\beta_{5,2}(K) &= 11\frac{\tilde{\alpha}_{5,2}(K)}{\tilde{\alpha}_{5,2}(3_1)}, & \beta_{6,8}(K) &= 17\frac{\tilde{\alpha}_{6,8}(K)}{\tilde{\alpha}_{6,8}(3_1)}, \\
\beta_{5,3}(K) &= \frac{\tilde{\alpha}_{5,3}(K)}{\tilde{\alpha}_{5,3}(3_1)}, & \beta_{6,9}(K) &= 271\frac{\tilde{\alpha}_{6,9}(K)}{\tilde{\alpha}_{6,9}(3_1)},
\end{aligned} \tag{4.16}$$

where $\tilde{\alpha}_{ij}(3_1)$ refers to the invariants corresponding to the trefoil knot. The compound Vassiliev invariants are then simply defined in such a way that they are products of primitive ones:

$$\begin{aligned}
\beta_{4,1} &= \beta_{2,1}^2, & \beta_{6,2} &= \beta_{3,1}^2, \\
\beta_{5,1} &= \beta_{2,1}\beta_{3,1}, & \beta_{6,3} &= \beta_{2,1}\beta_{4,2}, \\
\beta_{6,1} &= \beta_{2,1}^3, & \beta_{6,4} &= \beta_{2,1}\beta_{4,3}.
\end{aligned} \tag{4.17}$$

We will now compute the primitive Vassiliev invariants for torus knots invariants in such a normalization. From (4.9) follows:

$$\begin{aligned}
\beta_{2,1} &= \frac{1}{24}\,(n^2-1)\,(m^2-1), \\
\beta_{3,1} &= \frac{1}{144}\,n\,m\,(n^2-1)\,(m^2-1), \\
\beta_{4,2} &= \frac{1}{240}\,(n^2-1)\,(m^2-1)\,(9\,n^2\,m^2-n^2-m^2-1), \\
\beta_{4,3} &= \frac{1}{240}\,(n^4-1)\,(m^4-1),
\end{aligned}$$

$$\begin{aligned}
\beta_{5,2} =&\frac{1}{28800}\, n\, m\, (n^2-1)\, (m^2-1)\, \big(69\, n^2\, m^2 - 21\, \big(n^2+m^2\big) - 11\big),\\
\beta_{5,3} =&\frac{1}{57600}\, n\, m\, (n^2-1)\, (m^2-1)\, (11\, n^2\, m^2 + n^2 + m^2 - 9),\\
\beta_{5,4} =&\frac{1}{7200}\, n\, m\, (n^4-1)\, (m^4-1),
\end{aligned}$$

$$\begin{aligned}
\beta_{6,5} =&\frac{1}{2520}\, (n^2-1)\, (m^2-1)\, \big(516\, n^4\, m^4 - 289\, (n^2\, m^4 + n^4\, m^2)\\
&\qquad - 44\, n^2\, m^2 + 5\, (n^4+m^4) + 5\, (n^2+m^2) + 5\big),\\
\beta_{6,6} =&\frac{1}{12096}\, (n^2-1)\, (m^2-1)\, \big(53\, n^4\, m^4 - 101\, (n^2\, m^4 + n^4\, m^2)\\
&\qquad - 115\, n^2\, m^2 - 24\, (n^4+m^4) - 24\, (n^2+m^2) - 24\big),\\
\beta_{6,7} =&\frac{1}{10080}\, (n^2-1)\, (m^2-1)\, \big(419\, n^4\, m^4 + 209\, (n^2\, m^4 + n^4\, m^2)\\
&\qquad - n^2\, m^2 + 20\, (n^4+m^4) + 20\, (n^2+m^2) + 20\big),\\
\beta_{6,8} =&\frac{1}{25200}\, (n^2-1)\, (m^2-1)\, \big(13\, n^4\, m^4 + 13\, (n^2\, m^4 + n^4\, m^2)\\
&\qquad + 13\, n^2\, m^2 - 50\, (n^4+m^4) - 50\, (n^2+m^2) - 50\big),\\
\beta_{6,9} =&\frac{1}{5040}\, (n^2-1)\, (m^2-1)\, \big(31\, n^4\, m^4 + 31\, (n^2\, m^4 + n^4\, m^2)\\
&\qquad + 31\, n^2\, m^2 + 10\, (n^4+m^4) + 10\, (n^2+m^2) + 10\big).
\end{aligned} \tag{4.18}$$

In the following section we will describe some of the properties of these invariants.

5. Properties of the Vassiliev Invariants for Torus Knots

Given the Vassiliev invariants (4.18) the first question that one would like to answer is how many of them are needed to distinguish torus knots. It was shown in [18] that for torus knots of the form $(2, 2p+1)$ the Vassiliev invariants of second and third orders are enough. We will prove now that this result holds for arbitrary torus knots. In other words, we will prove that if two torus knots $\{n, m\}$ (with $(n, m) = 1$) and (n', m') (with $(n', m') = 1$) have the same first two Vassiliev invariants,

$$\begin{aligned}
\beta_{2,1}(n,m) &= \beta_{2,1}(n',m'),\\
\beta_{3,1}(n,m) &= \beta_{3,1}(n',m'),
\end{aligned} \tag{5.1}$$

then $(n',m') = (n,m)$, $(n',m') = (m,n)$, $(n',m') = (-n,-m)$, or $(n',m') = (-m,-n)$. First notice that from the explicit form of $\beta_{2,1}$ and $\beta_{3,1}$ in (4.18), one finds from (5.1):

$$m\,n = m'\,n', \qquad \text{and} \qquad n^2 + m^2 = {n'}^2 + {m'}^2. \tag{5.2}$$

Let γ be a rational number such that $m' = \gamma\,n$. From the first relation in (5.2) follows that also $m = \gamma\,n'$. Then, from the second relation in (5.2) one gets:

$$n^2(\gamma^2 - 1) = {n'}^2(\gamma^2 - 1), \tag{5.3}$$

which implies that either $n = \pm n'$ and γ arbitrary, or $\gamma = \pm 1$ and n and n' any. In the first of these two cases, after considering (5.2), one ends with either $n = n'$ and $m = m'$, or $n = -n'$ and $m = -m'$. In the second case, similarly, one concludes that either $m = n'$ and $n = m'$, or $m = -n'$ and $n = -m'$. The four possibilities correspond to the same torus knot and therefore we have proved that the first two Vassiliev invariants distinguish all torus knots.

This analysis shows that the first two Vassiliev invariants provide a one-to-one map between the set $\mathcal{T} = \{n, m \in \mathbf{Z} \mid (n,m) = 1,\ n > |m|\}$ and its image through the function $(\beta_{2,1}(n,m), \beta_{3,1}(n,m))$. From the form of $\beta_{2,1}$ and $\beta_{3,1}$ in (4.18) one finds that $\mathcal{T}$ gets mapped around a curve in the $(\beta_{2,1}, \beta_{3,1})$ plane whose asymptotic form (large values of $\beta_{2,1}$ and $\beta_{3,1}$) is:

$$\beta_{3,1}^2 = \frac{2}{3}\beta_{2,1}^3. \tag{5.4}$$

It is important to remark that our result for $\beta_{3,1}$ agrees with the Vassiliev invariant of order three for torus knots of the form $(2, 2p+1)$ presented in [18]: $v_3 = p^3 - p$. To verify this one must take into account that Vassiliev invariants are defined up to global factors and addition of a linear relation of lower order

Vassiliev invariants. It turns out that:

$$v_3 = 3(\beta_{3,1}(2, 2p+1) - \beta_{2,1}(2, 2p+1)) = p^3 - p. \tag{5.5}$$

Also, it would be interesting to study if for the lower order cases (two and three) the methods developped in [33] lead to results equivalent to ours.

Another important issue that we will address here is to work out which of the Vassiliev invariants which we have obtained contain the topological information. In other words, we will study at each order how many independent invariants are taking into account that linear combinations of lower-order invariants can be added. Certainly, at orders two and three there is a single one for each. At order four one finds that,

$$\beta_{4,2} = 4\beta_{4,3} + 12\beta_{2,1}^2 - \beta_{2,1}, \tag{5.6}$$

and therefore there is only one independent invariant. At order five one obtains,

$$\begin{aligned} \beta_{5,2} =& 6\beta_{5,3} + \frac{27}{5}\beta_{2,1}\beta_{3,1} - \frac{2}{5}\beta_{3,1}, \\ \beta_{5,3} =& \frac{3}{4}\beta_{5,3} + \frac{3}{10}\beta_{2,1}\beta_{3,1} - \frac{1}{20}\beta_{3,1}, \end{aligned} \tag{5.7}$$

and thus, again, there is only one independent invariant. At order six, however one finds that the number of independent invariants is two. Indeed, from the relations,

$$\begin{aligned} \beta_{6,5} =& \frac{58}{9}\beta_{6,9} - \frac{80}{3}\beta_{4,3} + \frac{41}{9}\beta_{2,1} - \frac{680}{3}\beta_{2,1}\beta_{4,3} + 5280\beta_{3,1}^2 - \frac{2080}{3}\beta_{2,1}^3, \\ \beta_{6,6} =& -\frac{5}{12}\beta_{6,9} - \frac{5}{3}\beta_{4,3} + \frac{1}{4}\beta_{2,1} - 10\beta_{2,1}\beta_{4,3} + 240\beta_{3,1}^2 - 40\beta_{2,1}^3, \\ \beta_{6,7} =& \frac{9}{2}\beta_{6,9} - 5\beta_{4,3} + \frac{1}{2}\beta_{2,1} + 432\beta_{3,1}^2 - 96\beta_{2,1}^3, \end{aligned} \tag{5.8}$$

and the fact that $\beta_{6,8}$ can not be written as a linear relation of the β_{ij} which appear on the right hand side of (5.8) one concludes that $\beta_{6,8}$ and $\beta_{6,9}$ can be taken as the independent invariants at order six. Notice that up to the order which we have studied the independent Vassiliev invariants for torus knots can be chosen for $i > 3$ as the ones associated to the Casimirs shown in Fig. 2. As discussed in [12], these Casimirs are the building blocks of the group factors shown in Fig. 1. One would like to know if the correspondence found up to order six holds in general.

The result obtained for $\beta_{2,1}$ allows to conclude that many torus knots are not Lissajous knots. It has been shown recently [34] that the Arf, Kervaire, or Robertello invariant for a Lissajous knot is zero. On the other hand, it is also known [15] that this invariant is just $\beta_{2,1}$ mod 2. Thus, if $\beta_{2,1}$ is odd for a torus knot $\{n, m\}$ one can state that it is not a Lissajous knot. From the formula for $\beta_{2,1}$ in (4.18) one can verify that indeed the trefoil knot ($n = 2$, $m = 3$) is not a Lissajous knot. Our result for the Vassiliev invariant $\beta_{2,1}$ in (4.18) allows to conclude in general that if $(n^2-1)(m^2-1)/24$ is odd the torus knot $\{n, m\}$ is not a Lissajous knot.

One of the most interesting property of the Vassiliev invariants β_{ij} is that they seem to be integer-valued. It was observed in [12] that there seemed to be a special normalization for the Vassiliev invariants such that they become integer-valued. One could say that such observation was not well-funded because after all in [12] only a finite set of knots were considered. However, in this paper we have studied an infinite subset of knots and one seems to find the same property. It is proved in the appendix that the β_{ij} in (4.18), which are polynomials in n and m, are integer-valued when n and m are coprime integers, $(n, m) = 1$, up to order four. For orders $i = 5, 6$ we have numerical evidence that this feature also hold but we do not have yet a proof. These polynomials share the property that there exist at least one value (then there are infinitely many) of the pair n, m, such that $(n, m) \neq 1$, for which β_{ij} is not integer-valued.

The invariants β_{ij} regarded as polynomials in n and m are very interesting by themselves from the point of view of number theory. These are polynomials which are integer-valued when $(n, m) = 1$ but fail for some n, m when $(n, m) \neq 1$. Given a polynomial with these features one can always construct a new one adding a new polynomial at most of the same degree with integer coefficients. Therefore, they are defined modulo the ring of polynomials in n and m with integer coefficients. The prototype for these polynomials is actually the one leading to the Gordian number of a torus knot. It has the form $(n-1)(m-1)/2$ (for $n, m > 0$) and according to Milnor's conjecture [35] corresponds to the uncrossing number of a torus knot $\{n, m\}$. Among the set of symmetric polynomials in n and m which only vanish for $n = 1$ and $m = 1$, the polynomial $(n-1)(m-1)/2$ is, up to a global sign, the only one which is integer-valued when $(n, m) = 1$ but it is not integer-valued for some n, m such that $(n, m) \neq 1$, modulo the ring of polynomials in n and m with integer coefficients.

Relative to the Gordian number, notice that our results show that $(n-1)(m-1)/2$ is not a Vassiliev invariant. If Milnor's conjecture holds this is consistent with the fact proven in [17,18] that the uncrossing number is not an invariant of finite type.

According to (4.18), the polynomials which seem to be relevant are symmetric polynomials in n and m which vanish for $n = \pm 1$ and $m = \pm 1$, and have the property described above: they are integer-valued when $(n, m) = 1$, and they fail to be an integer for some n, m such that $(n, m) \neq 1$. Actually, the set of polynomials in which one is interested is more restricted, for even orders they are invariant under $n \to -n$ (or $m \to -m$), while for odd orders they change sign under such a transformation. The properties and structure of this set of polynomials deserve to be studied from a general point of view.

We would like to end pointing out that one of the important consequences of knowing the explicit form of the Vassiliev invariants β_{ij} at least for torus knots is that it might shed light towards their general interpretation.

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APPENDIX

In this appendix we will prove that the Vassiliev invariants for torus knots as given in (4.18) are integer-valued for orders $i = 2, 3, 4$.

Proposition 1. If n and m are two coprime integers, $(n, m) = 1$, then $\beta_{2,1} = (n^2 - 1)(m^2 - 1)/24$ is integer-valued.

Proof: Let n and m be two integers such that $(n, m) = 1$. Being coprime there is at least one of them which is odd. Let this be n. Since the polynomial entering $\beta_{2,1}$ is symmetric in n and m this can be done without loss of generality. If n is odd one has that either $n - 1 = 0$ mod 2 and $n + 1 = 0$ mod 4, or $n + 1 = 0$ mod 2 and $n - 1 = 0$ mod 4. In either case, $n^2 - 1 = 0$ mod 8. If in addition n has the property of being $n = 1, 2$ mod 3 one has that either $n - 1 = 0$ mod 3, or $n + 1 = 0$ mod 3. In either case $n^2 - 1 = 0$ mod 3 and therefore $n^2 - 1 = 0$ mod 24, probing the proposition for this case. For the case which is left, $n = 0$ mod 3, one has that since $(n, m) = 1$, $m = 1, 2$ mod 3 which implies as before $m^2 - 1 = 0$ mod 3. Then, $(n^2 - 1)(m^2 - 1) = 0$ mod 24, and the propostion is proven. □

Proposition 2. If n and m are two coprime integers, $(n, m) = 1$, then $\beta_{3,1} = nm(n^2 - 1)(m^2 - 1)/144$ is integer-valued.

Proof: Let n and m be two integers such that $(n, m) = 1$. Being coprime, as in the previous proof one can always choose n to be odd without loss of generality. As shown there one has then that $n^2 - 1 = 0$ mod 8 and, if in addition $n = 1, 2$ mod 3, one has that $n^2 - 1 = 0$ mod 24. Let us consider the possible cases for m under this situation. We will consider the other situation (the one in which $n = 0$ mod 3 below). If m is odd and $m = 1, 2$ mod 3, the same arguments as before lead to $m^2 - 1 = 0$ mod 24, and therefore $(n^2 - 1)(m^2 - 1) = 0$ mod 24^2. If it where the case in which $m = 0$ mod 3 one would have $m(m^2 - 1) = 0$ mod 24, and therefore $m(n^2 - 1)(m^2 - 1) = 0$ mod 24^2. Finally, if m is even either $m + 1 = 0$

mod 3, or $m-1=0$ mod 3, being then $m(m^2-1)=0$ mod 6, and therefore $m(n^2-1)(m^2-1)=0$ mod 144. Let us consider now the second situation. If $n=0$ mod 3 one has that $n(n^2-1)=0$ mod 24. If m is odd, since $(n,m)=1$ one must have $m=1,2$ mod 3 and, as before, $m^2-1=0$ mod 24, obtaining then $n(n^2-1)(m^2-1)=0$ mod 24^2. If m is even, again $m(m^2-1)=0$ mod 6, and then $nm(n^2-1)(m^2-1)=0$ mod 144. This ends the proof of the proposition. □

Proposition 3. If n and m are two coprime integers, $(n,m)=1$, then $\beta_{4,3}=(n^4-1)(m^4-1)/240$ is integer-valued.

Proof: Let n and m be two integers such that $(n,m)=1$. Being coprime, as in the previous proof one can always choose n to be odd without loss of generality. As shown there one has then that $n^2-1=0$ mod 8. We will consider four possible situations:

a) $n=1,2$ mod 3 and $n=1,2,3,4$ mod 5. On the one hand, as in the previous proposition one has that $n^2-1=0$ mod 24. On the other hand, $n^2+1=0$ mod 2 and, if $n=2,3$ mod 5, $n^2+1=0$ mod 5, while if $n=1,4$ mod 5, $n^2-1=0$ mod 5. Then $(n^2-1)(n^2+1)=0$ mod 240.

b) $n=1,2$ mod 3 and $n=0$ mod 5. On the one hand, as before, $n^2-1=0$ mod 24, and $n^2+1=0$ mod 2. On the other hand, since $(n,m)=1$, $m=1,2,3,4$ mod 5. As in the previous situation, then $(m^2-1)(m^2+1)=0$ mod 5. Therefore, one concludes that $(n^2-1)(n^2+1)(m^2-1)(m^2+1)=0$ mod 240.

c) $n=0$ mod 3 and $n=1,2,3,4$ mod 5. In this case one has only that $n^2-1=0$ mod 8 and $n^2+1=0$ mod 2, while, again, $(n^2-1)(n^2+1)=0$ mod 5. As $(n,m)=1$ one must have $m=1,2$ mod 3, which implies that $m^2-1=0$ mod 3. Then, $(n^2-1)(n^2+1)(m^2-1)(m^2+1)=0$ mod 240.

d) $n=0$ mod 3 and $n=0$ mod 5. In this case one has only that $n^2-1=0$ mod 8 and $n^2+1=0$ mod 2. On the other hand, as $(n,m)=1$ one must have $m=1,2$ mod 3, which implies that $m^2-1=0$ mod 3, and $m=1,2,3,4$ mod 5, which implies that $(m^2-1)(m^2+1)=0$ mod 5. Then, $(n^2-1)(n^2+1)(m^2-1)(m^2+1)=0$ mod 240. □

Proposition 4. If n and m are two coprime integers, $(n, m) = 1$, then $\beta_{4,2} = (n^2 - 1)(m^2 - 1)(9n^2m^2 - n^2 - m^2 - 1)/240$ is integer-valued.

Proof: This proposition follows from equation (5.6) and Propositions 1 and 3. □

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